Multi-channel Queues with Setup Time
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Abstract
Many practical queuing situations with congestion control mechanism due to high throughput demands in telecommunication systems, computer network and production systems can be formulated as finite queues with setup time and state dependent arrivals. This chapter deals with computational scheme to compute the exact stationary queue length distribution. In this chapter an efficient iterative algorithm is developed for computing the stationary queue length distribution in M/G/K/N queues with setup time and arbitrary state dependent arrival rates. The overall computation of the algorithm is O(N2) in complexity. It can be of great use in application since it is easy to implement fast and quite accurate.

1. Introduction
The arrival occur according to Poisson process which depends on the number of customers in the system. We consider a M/G/K/N queue with state-dependent arrivals and set up time. Which was discussed earlier by Courtois and Georges (1971). The server has a set up time before serving the first customer who initializes a busy period which was best explained by Baker (1973) for the queue M/M/1 with exponential startup. Gordon and Newell (1967) also studied the queueing system with exponential servers. The service process is assumed to be independent of any process in system. The system can hold upto N customers including the one under service at any point of time. The service discipline is exhaustive and FCFS was studied by Shantikumar and Sumita (1985) of M/G/1/K queues with state dependent arrivals and FCFS/LCFS-p service disciplines.

This model has a wide range of application in telecommunication systems, production system and inventory control. ATM (Asynchronous Transfer Mode) technique is now broadly accepted for constructing high speed multimedia communication networks which was again analyzed by Skelly et al. (1993) and a Histogram based Model for Video Traffic Behaviour in an ATM multiplexer was developed. Reiser (1982) studied performance evaluation of data communication systems and due to the high throughput demands, these networks usually employ simplified and universal congestion control mechanism which are based on input rate enforcement in order to provide and maintain good quality of service (QoS) Schmidt and Compbell (1993) also studied Protocol Traffic Analysis with application for ATM switch Design which was brought forth by Keshav et al. (1995) through the study of an empirical evaluation of virtual circuit holding policies in Ip-over ATM Network.

It is believed that in a densely connected network the aggregated arrival process to the intermediate node can be approximated by a Poisson process suggested by Kee, and Towsley, (1986). A cell set up phase is generally needed before starting each busy period. This motivated up to study M(n)/G/K/N queue with set up time.

Mn/G/K/N queues have been given relatively little attention. Kijima and Makimoto (1992) give numerical algorithms to compute the quasi-stationary distribution and other characteristics in Mn/G/1/N queues and GI/M(n)/1/N queues by using Matrix-geometric method. Chaudhary, Gupta and Agarwal (1991) also examined computational analysis of distribution of numbers in system for M/G/1/N+1 and G/M/1/N+1 queues using roots. Gong et al (1992) also provide a numerical algorithm based on Matrix-geometric method of M/G/1 queues with state dependent arrivals. Recently, Yang proposes a new approach for computing the stationary queue length distribution in M(n)/G/1/N queues and GI/Mn/1/N queues. In this paper we develop an algorithm for computing the stationary queue length distribution in Mn/G/K/N queues with setup time by using the method of supplementary and variables. Buizen; (1973) also suggested computational Algorithms for closed Queueing Network with exponential servers. The rest of the paper is organized as follows. The section 2 we derive the system equations by using the method of supplementary variables. An interactive algorithm with overall. Computation O(N²) in complexity is developed for computing the stationary queue length distribution of Mn/G/1/N+1 with setup.

2. System Equations
Consider a M(n)/G/K/N queue with setup time described in section 1. Let λn be the arrival rate where there are n customers in the system. Since buffer size is N for n ≥ N, λn = 0. It is assumed that λn > 0 for 0 ≤ n ≤ N − 1. The probability density function (pdf) of the service time and its corresponding Laplace transform (L.T.) are denoted by b (.) and B*(s), respectively. We denote the mean of the service time by κµ. The pdf of the service time is denoted by a (.) with L.T. A*(s) and mean v. Let Q(t) be the number of customers in the system at time t. We define the
supplementary variable $U(t)$ as the remaining service time or the remaining setup time at time $t$. Let
\[ R(t) = \begin{cases} 
1 & \text{server is busy or stays idle at time } t; \\
0 & \text{server is setting up at time } t.
\end{cases} \]

Clearly, the process $(\{Q(t), R(t), U(t)\}; t \geq 0)$ is a Markov chain. Define the steady state joint density functions of $(Q(t), R(t), U(t)); t \geq 0)$ as
\[ f_n(u)\Delta u = \lim_{\Delta u \to 0} \Pr\{Q(t) = n, R(t) = 1, u < U(t) < u + \Delta u\}; \]
\[ g_n(u)\Delta u = \lim_{\Delta u \to 0} \Pr\{Q(t) = n, R(t) = 0, u < U(t) < u + \Delta u\}. \]

Let $Q$ be the number of customers in the system in steady state. Then, the stationary queue length distribution is given by $P(Q=n), 0 \leq n < \infty$. By infinitesimal argument (idea given by Taylor and Karlin, 1994), we have following steady state equations:
\[ \lambda_0 P(Q=0) = f_0(0), \]
\[ \frac{df_1(u)}{du} = -\lambda_1 f_1(u) + f_2(0)b(u) + g_1(0)b(u), \]
\[ f_n(0) = +\lambda_{n-1}f_{n-1}(u) - \lambda_n f_n(u) + f_{n+1}(0)b(u) + g_n(0)b(u), 2 \leq n \leq N - 1, \]
\[ \frac{dg_1(u)}{du} = \lambda_0 P(Q=0)a(u) - \lambda_1 g_1(u), \]
\[ \frac{dg_n(u)}{du} = \lambda_{n-1}g_{n-1}(u) - \lambda_ng_n(u), 2 \leq n \leq N - 1 \]
\[ \frac{dg_n(u)}{du} = \lambda_{n-1}g_{n-1}(u), \]

Denote
\[ p_n^*(s) = \int_0^\infty e^{-su}f_n(u)du, \quad n = 1, 2, \ldots, N, \]
\[ q_n^*(s) = \int_0^\infty e^{-su}g_n(u)du, \quad n = 1, 2, \ldots, N, \]
And $p_0^*(0) = P(Q = 0)$. Since $p_0^*(0)$ and $p_n^*(0) + q_n^*(0) = P(Q = n)$ for $1 \leq n \leq N$ are the stationary queue length distribution, our objective is to determine $\{p_0^*(0), p_n^*(0), q_n^*(0), 1 \leq n \leq N\}$. Taking the Laplace transform of (1), we have
\[ \lambda_0 p_0^*(0) = f_0(0), \]
\[ (\lambda_1 - s)p_1^*(s) = B(s)f_2(0) + B(s)g_1(0) - f_1(0), \]
\[ (\lambda_n - s)p_n^*(s) = \lambda_{n-1}p_{n-1}^*(s) + B(s)g_n(0) - f_n(0), \]
\[ 2 \leq n \leq N - 1, \]
\[ -sp_n^*(s) = \lambda_{n-1}p_{n-1}^*(s) + B(s)g_n(0) - f_n(0), \]
\[ \ldots \]
\[ (\lambda_1 - s)q_1^*(s) = \lambda_0 A(s)p_0^*(0) - g_1(0), \]
\[ (\lambda_n - s)q_n^*(s) = \lambda_{n-1}q_{n-1}^*(s) - g_n(0), 2 \leq n \leq N - 1, \]
\[ -sq_n^*(s) = \lambda_{n-1}q_{n-1}^*(s) - g_n(0). \]

By substituting $s=0$ into equation (2), we can have following lemma which gives expression of $f_i(0)$ and $g_i(0)$ in terms of $p_0^*(0)$'s and $q_0^*(0)$'s after some algebraic manipulations.

Lemma 1
\[ f_1(0) = \lambda_0 p_0^*(0), \]
\[ f_n(0) = \lambda_{n-1}p_{n-1}^*(0) + q_{n-1}^*(0), 2 \leq n \leq N, \]
\[ g_1(0) = \lambda_0 p_0^*(0) - \lambda_1 q_1^*(0), \]
\[ g_n(0) = \lambda_{n-1}q_{n-1}^*(0) - \lambda_n q_n^*(0), 2 \leq n \leq N - 1, \]
\[ g_n(0) = \lambda_{n-1}q_{n-1}^*(0) - \lambda_n q_n^*(0). \]

\[ \ldots \]

We can eliminate $f_i(0)$'s and $g_i(0)$'s in (2) by using Lemma 1.
\[ (\lambda_1 - s)p_1^*(s) = \lambda_0 B(s)p_0^*(0) + \lambda_0 (B(s) - 1)p_0^*(0), \]
\[ (\lambda_n - s)p_n^*(s) = \lambda_{n-1}p_{n-1}^*(s) - p_{n-1}^*(0) + \lambda_n B(s)p_0^*(0) + \lambda_n (B(s) - 1)q_{n-1}^*(0), \]
\[ -sp_n^*(s) = \lambda_{n-1}p_{n-1}^*(s) - p_{n-1}^*(0) + \lambda_n p_{n-1}^*(0) + \lambda_n q_{n-1}^*(0), \]
\[ \lambda_1 - sq_1^*(s) = \lambda_0 A(s) - 1)p_0^*(0) + \lambda_1 q_1^*(0), \]
\[ (\lambda_n - s)q_n^*(s) = \lambda_{n-1}q_{n-1}^*(s) - q_{n-1}^*(0) + \lambda_n q_{n-1}^*(0), \]
\[ -sq_n^*(s) = \lambda_{n-1}q_{n-1}^*(s) - q_{n-1}^*(0). \]

For $2 \leq n \leq N - 1$, setting $s = \lambda_i$ into the $i$th and the $(N+1)$th equations in (4) for $i = 1, 2, \ldots, N - 1$ gives
\[ p_1^*(0) = \lambda_0 \frac{1 - B(s)\lambda_i}{\lambda_1 B(s)\lambda_i} p_0^*(0), \]
\[ q_1^*(0) = \lambda_0 \frac{1 - A(s)\lambda_i}{\lambda_1} p_0^*(0), \]
\[ p_n^*(0) = \lambda_0 \frac{1 - B(s)\lambda_i}{\lambda_1 B(s)\lambda_i} p_{n-1}^*(0), \]
\[ q_n^*(0) = \lambda_0 \frac{1 - A(s)\lambda_i}{\lambda_1} p_{n-1}^*(0). \]
The desired result follows immediately by using (7).
\[ p_n^*(0) = \frac{x_n(0)}{1 + \sum_{n=1}^{N} [x_n(0) + y_n(0)]} \]
\[ q_n^*(0) = \frac{y_n(0)}{1 + \sum_{n=1}^{N} [x_n(0) + y_n(0)]} \]
for \( 1 \leq n \leq N \). The next lemma provides a formula of \( x_n(0) + y_n(0) \) in terms of \( [x_n(0), y_n(0), 1 \leq n \leq N - 1] \).

**Lemma 2**

\[ x_N(0) + y_N(0) = (k\mu + v)p_0 + \sum_{n=1}^{N-1} (\lambda_n - k\mu - 1) [x_n(0) + y_n(0)] \]

**Proof**

Adding equations in (4), we have
\[ -s \sum_{n=1}^{N} [p_n^*(0) + q_n^*(0)] = -s \sum_{n=1}^{N} \left[ \frac{\lambda_0[A^*(s) - 1]p_0^*(0) + \lambda_1 q_1^*(0)}{\lambda_1 - s} \right] + \lambda_n B^*(\lambda_n) p_n^*(s) + \lambda_n q_n^*(s) \]
for \( 2 \leq n \leq N - 1 \). Note that \( p_n^*(s) \) and \( q_n^*(s) \) for \( n \geq 1 \) in (6) are well-defined at \( s = \lambda_n \) because both numerator and denominator of \( p_n^*(s) \) and \( q_n^*(s) \) for \( n \geq 1 \) have zero at \( s = \lambda_n \). We denote (5) and (6) as the system equations. An iterative algorithm will be developed for computing the stationary queue length distributions
\[ \{ p_n^*(0), p_n^*(0) + q_n^*(0), 1 \leq n \leq N \} \]
for these equations.

3. **The Algorithm**

In this section, we develop an efficient scheme for computing the stationary queue length distribution with overall computation \( O(N^2) \) in complexity.

From the system equations (5) and (6), there exist \( x_n(s) \) and \( y_n(s) \) such that
\[ p_n^*(s) = x_n(s)p_n^*(0), \]
\[ q_n^*(s) = y_n(s)p_n^*(0), \]
for \( n = 1, 2, ..., N \). By the normalization condition
\[ p_0^*(0) + \sum_{n=1}^{N} \left[ p_n^*(0) + q_n^*(0) \right] = 1 \]
and (7),

Thus, from (7) and (8)
\[ x_1(0) = \frac{\lambda_0 \left[ 1 - x_0(\lambda_1) \right]}{\lambda_1 x_0(\lambda_1)} \]
\[ y_1(0) = \frac{\lambda_0 \left[ 1 - y_0(\lambda_1) \right]}{\lambda_1} \]
\[ x_n(0) = \frac{\lambda_n - 1}{\lambda_n} y_{n-1}(0) \]
\[ y_n(0) = \frac{\lambda_n - 1}{\lambda_n} \left[ y_{n-1}(0) - y_{n-1}(\lambda_n) \right] \]
\[ x_1(s) = \frac{\lambda_0 [x_0(s) - 1] + \lambda_1 x_0(s)x_0(0)}{\lambda_1 - s} \quad \text{...(11)} \]
\[ y_1(s) = \frac{\lambda_0 [y_0(s) - 1] + \lambda_1 y_0(s)}{\lambda_1 - s} \]
\[ x_n(s) = \frac{\lambda_n - 1}{\lambda_n} [x_{n-1}(s) - x_{n-1}(0)] + \lambda_n x_0(s)x_n(0) \]
\[ y_n(s) = \frac{\lambda_n - 1}{\lambda_n} \left[ y_{n-1}(s) - y_{n-1}(0) \right] + \lambda_n y_0(s) \]

for \( 2 \leq n \leq N - 1 \). Observe that in order to obtain \( x_n(0), y_n(0), 1 \leq n \leq N - 1 \), we still need to evaluate \( x_{n-1}(\lambda_n), y_{n-1}(\lambda_n), 1 \leq n \leq N - 1 \). Let \( w_i(s) = [d^i w(s)]/[ds]^i \). By some algebraic manipulations, one can have,

\[ x_1^{(i)}(s) = \frac{\lambda_0 + \lambda_1 x_1(0)}{\lambda_1 - s} \]
\[ y_1^{(i)}(s) = \frac{\lambda_0 y_0^{(i)}(s) + i y_1^{(i-1)}(s)}{\lambda_1 - s} \]
\[ x_n^{(i)}(s) = \frac{\lambda_n - 1}{\lambda_n} [x_{n-1}(s) - x_{n-1}(0)] + \lambda_n x_0(s)x_n(0) \]
\[ y_n^{(i)}(s) = \frac{\lambda_n - 1}{\lambda_n} \left[ y_{n-1}(s) - y_{n-1}(0) \right] + \lambda_n y_0(s) \]

for \( 2 \leq n \leq N - 1 \), \( i \geq 1 \).

Therefore, if \( \lambda_k \neq \lambda_n \),

\[ x_1(\lambda_k) = \frac{\lambda_0 [x_0(\lambda_k) - 1] + \lambda_1 x_0(\lambda_k)x_1(0)}{\lambda_1 - \lambda_k} \]
\[ y_1(\lambda_k) = \frac{\lambda_0 [y_0(\lambda_k) - 1] + \lambda_1 y_1(0) + \lambda_n y_0(\lambda_k)x_n(0)}{\lambda_1 - \lambda_k} \]
\[ x_n(\lambda_k) = \frac{\lambda_n - 1}{\lambda_n} \left[ x_{n-1}(\lambda_k) - x_{n-1}(0) \right] + \lambda_n x_0(\lambda_k)x_n(0) \]
\[ y_n(\lambda_k) = \frac{\lambda_n - 1}{\lambda_n} \left[ y_{n-1}(\lambda_k) - y_{n-1}(0) \right] + \lambda_n y_0(\lambda_k) \]

for \( 2 \leq n \leq N - 1 \), \( i \geq 1 \).

However, it is not necessary to evaluate all \( x_n^{(i)}(\lambda_k), y_n^{(i)}(\lambda_k), 0 \leq n \leq N - 1, 0 \leq i \leq N - 1 \). 0 \leq k \leq N - 1 \) to calculate \( x_{n}(0), y_{n}(0), 1 \leq n \leq N - 1 \). An efficient scheme is developed in the following.

For simplicity, we may assume that the arrival rates can be divided into \( m \) groups based on \( m + 1 \) threshold values \( N_0 = 0 < N_1 < N_2 < \ldots < N_m = N \) such that
for $i \leq 0$, where $\lambda_{N_{i_1}} \neq \lambda_{N_{i_2}}$ if $i_1 \neq i_2$. This assumption is very practical, although it is not hard to modify the following algorithm to the general cases.

Let $l(k) = \max \left\{ 1, N_{i_k} \right\}$ such that

$$N_{i_k} \leq k \leq N_{i_{k+1}}$$

is the least positive number with

$$l(l(k)) = \lambda_k.$$ Denote $L_n(k)$ as the number of $\lambda_i$ such that $\lambda_i = \lambda_k$ for $n + 1 \leq i \leq k$, that is, $L_n(k) = k - \max \left\{ n+1, l(l(k)) \right\}$ for $0 \leq n \leq k - 1$ and $2 \leq k \leq N - 1$. From above definitions, we immediately have following lemma:

**Lemma 3**

For $0 \leq n \leq k - 1, 2 \leq k \leq N - 1,$

1. If $L_n(k) \geq 1$, $\lambda_i = \lambda_k$ and $L_n(k) = \ln(k+1)+1$.
2. If $\lambda_n = \lambda_{n+1} = \lambda_{n+2}$, $L_n(k) = L_{n+3}(k)$.
3. If $\lambda_n = \lambda_{n+1}, L_n(k) = L_{n+4}(k)-1$.

**Lemma 4**

If $1 \leq L_n(k) + l(l(k)) \leq k - 1, 2 \leq k \leq N - 1,$

then $X_0^{l(l(k))} = X_0^{l(l(k)) \left( \lambda_1 \right)}$.

**Proof:**

$I = L_n(k) + l(l(k)) = k - \max \left\{ n, l(l(k)) \right\}$, $l(l(k)) \leq k$.

On the other hand, $l = L_n(k) + l(l(k)) \geq l(l(k))$. Thus, by assumption on the arrival rates, $\lambda_1 = \lambda_k$. Since we always have $l(l(k)) \geq 1$,

$L_{n+1}(l(l(k))) = 1 - \max \left\{ 1, l(l(k)) \right\} = 1 - \max \left\{ 1, 0 \right\}$.

We arrive at the desired result.

Using Lemma 3 and lemma 4, (13) and (14) can be written as following:

For $\lambda_n \neq \lambda_{n+1}, 1 \leq n \leq k - 1, 2 \leq k \leq N - 1$.

if $L_n(k) = 0$

$$y_n^{[L_n(k)]}(\lambda_k) = \frac{-\lambda_{n-1} \left( y_{n-1}^{[L_{n-1}(k)]}(\lambda_k) - y_{n-1}(0) \right) + \lambda_n y_0(0)}{\lambda_n - \lambda_k}$$

if $L_n(k) \geq 1$,

$$x_1^{[L_1(k)]}(\lambda_k) = \frac{\left[ \lambda_0 + \lambda_1 x_1(0) \right] x_0^{[L_0(k)]}(\lambda_k)}{\lambda_1 - \lambda_k}$$

$$y_1^{[L_1(k)]}(\lambda_k) = \frac{\left[ \lambda_0 + \lambda_1 x_1(0) \right] x_0^{[L_0(k)]}(\lambda_k) + \lambda_n y_0(0)}{\lambda_n - \lambda_k}$$

$$x_1^{[L_1(k)]}(\lambda_k) = \frac{\lambda_{n-1} \left( x_{n-1}(0) \right) + \lambda_n x_n(0)}{\lambda_n - \lambda_k}$$

$$Y_n(0) = \frac{\left[ \lambda_0 + \lambda_1 x_1(0) \right] x_0^{[L_0(k)]}(\lambda_k) + \lambda_n y_0(0)}{\lambda_n - \lambda_k}$$

Using the following algorithm, we use $x_n(k), y_n(k)$ to store $X_n^{[L_n(k)]}, Y_n^{[L_n(k)]}$ respectively.

**Algorithm**

**Step 1:**

Given $\lambda_n, 0 \leq n \leq N - 1$ and $k, n$, $\lambda_{ik}$,

For $1 \leq k \leq N - 1$, find $i_k$

$$N_{ik_1} = k \leq N_{ik_2}$$

such that and $l(k) = \max \left\{ 1, N_{ik_2} \right\}$.

**Step 2:**

For $k = 2, 3, \ldots, N - 1$, do

(a) For $n = 0, 1, \ldots, k - 1$, do.....(15)
\[ L_n(k) \leftarrow k - \max \{n+1, l(k)\}. \]
If \( n \geq 1 \), \( l \leftarrow L_n(k) + l(k) \).
If \( n = 0 \) then
\[ x_n(k) \leftarrow B^*\{L_n(k)\}(\lambda_k) \]
\[ y_n(k) \leftarrow A^*\{L_n(k)\}(\lambda_k) \]
Else if \( \lambda_n = \lambda_k \) then
\[ x_n(k) \leftarrow -\frac{[\lambda_0 + \lambda_1x_0(0)x_0(k)]}{L_1(k) + 1} \]
\[ y_n(k) \leftarrow -\frac{\lambda_n - \lambda_k}{L_1(k) + 1} \]
Else if \( L_n(k) \geq 1 \) then
\[ x_1(k) \leftarrow \frac{\lambda_n x_n(0) + \lambda_n - \lambda_k}{\lambda_n - \lambda_k} \]
\[ y_1(k) \leftarrow \frac{\lambda_n y_n(0) + \lambda_n - \lambda_k}{\lambda_n - \lambda_k} \]
End (n).

Step 3:
\[ x_N(0) + y_N(0) \leftarrow \lambda_0 (k\mu + v) + \sum_{k=1}^N (\lambda_k k\mu - 1) [x_k(0) + y_k(0)] \]

Step 4:
\[ P(Q = 0) \leftarrow \frac{1}{1 + \sum_{k=1}^N [x_k(0) + y_k(0)]} \]

Step 5:
\[ P(Q = k) \leftarrow \frac{x_k(0) + y_k(0)}{L_k(k) + 1} \]

Further we can use the above algorithm to obtain the stationary queue length distribution for \( M(n)/G/k/N \) queues by letting \( v = 0 \) and \( A^*(s) = 1 \).

4. Numerical Result

In this section we use the algorithm to obtain the stationary queue length distribution in four \( M(n)/G/k/N \) queuing systems with or without setup times. The results for \( M(n)/E_k/k/N \) with setup time for \( N \in \{10, 20, 30\} \). The arrival rate is \( \lambda_n = N - n \) if there are \( n \) customers in the system for \( 0 \leq n \leq N \). The other parameters for the algorithm are mean service time \( \mu = \frac{1}{15} \), the mean setup time \( v = 0 \) and \( A^*(s) = 1 \). It is compared with the result given by Kijima and Makimoto (1992).

Numerical result for \( M(n)/E_k/k/N \) queues with set up time and \( N \in \{10, 20, 3\} \). The arrival and the mean service time are the same while two types of setup time are chosen to test. The first one is exponential distribution with mean 1/30 and the other one is Erlang distribution with 2 phases and mean 1/30.

The results are compared by result given by Gong et al. (1992).

5. Conclusion

Two types of set up time are considered in this. The first queue has an exponential set up time, with mean 0.1 while the second queue has deterministic set up time 0.1. In conclusion, the algorithm is powerful for general cases and it is easy to implement, fast and quite accurate.
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